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Ramanujan's Theory of Elliptic Functions to the Cubic Base

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Abstract. In this article, we develop Ramanujan's theory of elliptic functions to the cubic base using Jacobi's theta functions. Our new approach does not involve the theta series discovered by J. M. Borwein and P. B. Borwein, Goursat's transformation formulas for the hypergeometric series, analogue of Gauss' AGM and the theory of modular forms.

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1. Introduction

A well known identity of Jacobi [12, p. 90, (14)] states that if $|q| < 1$, then

$$\prod_{k=1}^{\infty}\left(1+q^{2k-1}\right)^8 = \prod_{k=1}^{\infty}\left(1-q^{2k-1}\right)^8 + 16q\prod_{k=1}^{\infty}\left(1+q^{2k}\right)^8. \tag{1.1}$$

Using Jacobi's triple product identity (see [7, Theorem 3.2] or (2.4)), one can show that (1.1) is equivalent to the identity

$$\left(\sum_{m,n=-\infty}^{\infty} q^{m^2+n^2}\right)^2 = \left(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{m^2+n^2}\right)^2 + \left(\sum_{m,n=-\infty}^{\infty} q^{(m+1/2)^2+(n+1/2)^2}\right)^2. \tag{1.2}$$

Dedicated to Professor J.-P. Serre on the occasion of his 99th birthday.

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Around 1991, J.M. Borwein and P.B. Borwein [4] discovered a cubic analogue of (1.2) given by

$$\left(\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}\right)^3 = \left(\sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}\right)^3 + \left(\sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}\right)^3 \tag{1.3}$$

where $\omega = e^{2\pi i/3}$. The three theta series in (1.3) are now known as Borweins' theta series. The first proof of (1.3) is given by the Borweins [4]. For other proofs of (1.3) and its generalizations, see D. Schultz [17], R. Chapman [10] and J.M. Borwein, F.G. Garvan and M. Hirschhorn [6].

The Borweins theta series in (1.3) are usually denoted by $a(q), b(q)$ and $c(q)$ respectively. However, in this article, we will reserve the letters $a = a(\tau)$, $b = b(\tau)$ and $c = c(\tau)$ for functions which will appear naturally in our derivations of certain identities associated with Jacobi's theta functions.

Around 1994, using Borweins' theta series as their starting point, B.C. Berndt, S. Bhargava and F.G. Garvan (see [3] and [2, Chapter 33]) succeeded in developing Ramanujan's theory of elliptic functions to the cubic base, a theory that was briefly mentioned by Ramanujan in [15]. In their work, the following transformation formula of Goursat [2, Corollary 2.4] plays an important role:

$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \left(\frac{1-r}{1+2r}\right)^3\right) = (1+2r)\, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; r^3\right), \tag{1.4}$$

where

$${}_2F_1\left(c, d; e; u\right) = \sum_{j=0}^{\infty} \frac{(c)_j (d)_j}{(e)_j} \frac{u^j}{j!},$$

with

$$(\ell)_n = \prod_{k=1}^{n} (\ell + k - 1).$$

In [2, p. 97], Berndt remarked that the Borweins deduced (1.4) in connection with their cubic analogue of the arithmetic–geometric mean [4] while his approach with Bhargava and Garvan depended upon prior knowledge of the identity and differential equations and that both approaches are not completely satisfactory.

In this article, we present an approach to Ramanujan's theory of elliptic functions to the cubic base without reference to Goursat's formula and cubic analogue of the arithmetic–geometric mean.

2. Important Facts About the Jacobi Theta Function $\vartheta_1(u|\tau)$

The Jacobi theta function $\vartheta_1(u|\tau)$ is defined by

$$\vartheta_1(u|\tau) = -i \sum_{j=-\infty}^{\infty} (-1)^j q^{(j+1/2)^2} e^{(2j+1)iu}, \tag{2.1}$$

where $q = e^{\pi i \tau}$. It satisfies two basic transformation formulas

$$\vartheta_1(u+\pi|\tau) = -\vartheta_1(u|\tau) \tag{2.2}$$

and

$$\vartheta_1(u+\pi\tau|\tau) = -q^{-1}e^{-2iu}\vartheta_1(u|\tau) \tag{2.3}$$

which follows directly from the definition (2.1) of $\vartheta_1(u|\tau)$.

An identity associated with $\vartheta_1(u|\tau)$ known as the Jacobi triple product identity expresses $\vartheta_1(u|\tau)$ as an infinite product as follows (see for example [7, Theorem 3.2]):

$$\vartheta_1(u|\tau) = 2q^{1/4}\sin u \prod_{k=1}^{\infty}(1-q^{2k})(1-q^{2k}e^{2iu})(1-q^{2k}e^{-2iu}). \tag{2.4}$$

By applying logarithmic differentiation to (2.4), we get

$$\frac{\vartheta_1'}{\vartheta_1}(u|\tau) = \cot u + 4\sum_{j=1}^{\infty}\frac{q^{2j}}{1-q^{2j}}\sin 2ju, \tag{2.5}$$

where we have used

$$\frac{f'}{f}(u) = \frac{f'(u)}{f(u)}.$$

Expanding (2.5), we find that

$$\frac{\vartheta_1'}{\vartheta_1}(u|\tau) = \frac{1}{u} + \sum_{j=1}^{\infty}(-1)^j\frac{2^{2j}}{(2j)!}B_{2j}L_{2j}u^{2j-1}, \tag{2.6}$$

where

$$L_{2j} = L_{2j}(\tau) = 1 - \frac{4j}{B_{2j}}\sum_{\ell=1}^{\infty}\frac{\ell^{2j-1}q^{2\ell}}{1-q^{2\ell}} \tag{2.7}$$

with the Bernoulli numbers B_k defined by

$$\frac{1}{e^{2it}-1} = \sum_{k=0}^{\infty}\frac{B_k}{k!}(2it)^{k-1}.$$

We will also need the following transformation formulas which are consequences of (2.2) and (2.3):

$$\frac{\vartheta_1'}{\vartheta_1}(u+\pi|\tau) = \frac{\vartheta_1'}{\vartheta_1}(u|\tau) \tag{2.8}$$

and

$$\frac{\vartheta_1'}{\vartheta_1}(u+\pi\tau|\tau) = \frac{\vartheta_1'}{\vartheta_1}(u|\tau) - 2i. \tag{2.9}$$

Lastly, we record the following identities which we will need in our subsequent sections. These identities are consequences of (2.4):

$$\vartheta_1'(0|\tau) = 2q^{1/4}\prod_{k=1}^{\infty}\left(1-q^{2k}\right)^3 \tag{2.10}$$

$$\vartheta_1(\pi/3|\tau) = \sqrt{3}q^{1/4}\prod_{k=1}^{\infty}\left(1-q^{6k}\right) \tag{2.11}$$

and

$$\vartheta_1(\pi\tau|3\tau) = iq^{-1/4}\prod_{k=1}^{\infty}\left(1-q^{2k}\right). \tag{2.12}$$

3. Identities Involving $L_4(\tau)$ and $L_6(\tau)$

In this section, we establish the following identity:

Theorem 3.1. *Let $q = e^{\pi i\tau}$ with $Im\,\tau > 0$ and $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. Let*

$$a = a(\tau) = 1 + 6\sum_{j=1}^{\infty}\left(\frac{j}{3}\right)\frac{q^{2j}}{1-q^{2j}} \tag{3.1}$$

and

$$b = b(\tau) = \prod_{k=1}^{\infty}\frac{(1-q^{2k})^3}{1-q^{6k}}. \tag{3.2}$$

Then

$$\frac{\vartheta_{1'}}{\vartheta_1}(u+\pi/3|\tau) - \frac{\vartheta_{1'}}{\vartheta_1}(u-\pi/3|\tau) + \frac{2}{\sqrt{3}}b\frac{\vartheta_1^2(u|\tau)}{\vartheta_1(u+\pi/3|\tau)\vartheta_1(u-\pi/3|\tau)} = \frac{2}{\sqrt{3}}a, \tag{3.3}$$

We observe that in Theorem 3.1, our functions $a(\tau)$ and $b(\tau)$ appear naturally as coefficients in the identity (3.3).

Proof. From (2.8) and (2.9), we deduce that

$$\frac{\vartheta_1'}{\vartheta_1}(u+\pi/3|\tau) - \frac{\vartheta_1'}{\vartheta_1}(u-\pi/3|\tau)$$

is an elliptic function with periods π and $\pi\tau$ having simple poles at $u = \pi/3$ and $u = -\pi/3$. The function

$$\frac{\vartheta_1^2(u|\tau)}{\vartheta_1(u+\pi/3|\tau)\vartheta_1(u-\pi/3|\tau)}$$

is also an elliptic function with periods π and $\pi\tau$ having simple poles at $u = \pi/3$ and $u = -\pi/3$. Therefore, there exist constants ξ and χ independent of u such that

$$\frac{\vartheta_1'}{\vartheta_1}(u+\pi/3|\tau) - \frac{\vartheta_1'}{\vartheta_1}(u-\pi/3|\tau) + \xi\frac{\vartheta_1^2(u|\tau)}{\vartheta_1(u+\pi/3|\tau)\vartheta_1(u-\pi/3|\tau)} = \chi. \tag{3.4}$$

Setting $u = 0$ implies that

$$\chi = \frac{\vartheta_1'}{\vartheta_1}(\pi/3|\tau) - \frac{\vartheta_1'}{\vartheta_1}(-\pi/3|\tau) = \frac{2}{\sqrt{3}}a.$$

Next, we rewrite (3.4) as

$$\begin{aligned}&\vartheta_1'(u+\pi/3|\tau)\vartheta_1(u-\pi/3|\tau) - \vartheta_1'(u-\pi/3|\tau)\vartheta_1(u+\pi/3|\tau) + \xi\vartheta_1^2(u|\tau)\\ &\quad= \frac{2}{\sqrt{3}}a\vartheta_1(u+\pi/3|\tau)\vartheta_1(u-\pi/3|\tau).\end{aligned} \tag{3.5}$$

Letting $u = \pi/3$ in (3.5) and using (2.10) and (2.11), we deduce that

$$\xi = \frac{\vartheta_1'(0|\tau)}{\vartheta_1(2\pi/3)} = \frac{2}{\sqrt{3}}b,$$

where b is given by (3.2). This completes the proof of (3.3). □

Next, we establish the following identity

Theorem 3.2. *Let* $q = e^{\pi i\tau}$ *with* $Im\,\tau > 0$. *Then*

$$\begin{aligned}&\vartheta_1^3(z|\tau) - \vartheta_1^3(z+\pi/3|\tau) - \vartheta_1^3(z-\pi/3|\tau)\\ &\qquad= 3\frac{a}{b}\vartheta_1(z|\tau)\vartheta_1(z+\pi/3|\tau)\vartheta_1(z-\pi/3|\tau).\end{aligned} \tag{3.6}$$

Identity (3.6) can be found in Z.G. Liu's article [13, (5.12)]. We give a proof of this identity here. We now use (3.3) to give a proof of (3.6).

Proof. Rewrite (3.3) as

$$\begin{aligned}&\frac{\vartheta_1'}{\vartheta_1}(u+\pi/3|\tau) - \frac{\vartheta_1'}{\vartheta_1}(u-\pi/3|\tau)\\ &\qquad+ \frac{2}{\sqrt{3}}b\frac{\vartheta_1^3(u|\tau)}{\vartheta_1(u|\tau)\vartheta_1(u+\pi/3|\tau)\vartheta_1(u-\pi/3|\tau)} = \frac{2}{\sqrt{3}}a.\end{aligned} \tag{3.7}$$

By replacing u by $u-\pi/3$ and $u-2\pi/3$ in (3.7) and using (2.8) and (2.9), we deduce that

$$\frac{\vartheta_1'}{\vartheta_1}(u+\pi/3|\tau)-\frac{\vartheta_1'}{\vartheta_1}(u-\pi/3|\tau) \\ -\frac{2}{\sqrt{3}}b\frac{\vartheta_1^3(u-\pi/3|\tau)}{\vartheta_1(u|\tau)\vartheta_1(u+\pi/3|\tau)\vartheta_1(u-\pi/3|\tau)}=\frac{2}{\sqrt{3}}a \tag{3.8}$$

and

$$\frac{\vartheta_1'}{\vartheta_1}(u-\pi/3|\tau)-\frac{\vartheta_1'}{\vartheta_1}(u|\tau) \\ -\frac{2}{\sqrt{3}}b\frac{\vartheta_1^3(u+\pi/3|\tau)}{\vartheta_1(u|\tau)\vartheta_1(u+\pi/3|\tau)\vartheta_1(u-\pi/3|\tau)}=\frac{2}{\sqrt{3}}a. \tag{3.9}$$

Adding (3.7), (3.8) and (3.9) and simplifying, we deduce that

$$\frac{2}{\sqrt{3}}b\left(\vartheta_1^3(u|\tau)-\vartheta_1^3(u-\pi/3|\tau)-\vartheta_1^3(u+\pi/3|\tau)\right) \\ =\frac{6}{\sqrt{3}}a\vartheta_1(u|\tau)\vartheta_1(u+\pi/3|\tau)\vartheta_1(u-\pi/3|\tau),$$

and the proof of (3.6) is complete.

□

Let

$$R:=R(u|\tau)=\frac{\vartheta_1(u|\tau)}{\vartheta_1(u-\pi/3|\tau)} \tag{3.10}$$

and

$$S:=S(u|\tau)=-\frac{\vartheta_1(u+\pi/3|\tau)}{\vartheta_1(u-\pi/3|\tau)}. \tag{3.11}$$

We may then rewrite (3.6) as

$$R^3+S^3+3\frac{a}{b}RS=1. \tag{3.12}$$

Differentiating both sides of (3.12) with respect to u, we deduce that

$$R'\left(aS+bR^2\right)=-S'\left(aR+bS^2\right). \tag{3.13}$$

Now, from (3.3), we find that

$$\frac{S'}{S}=\frac{\vartheta_1'}{\vartheta_1}(u+\pi/3|\tau)-\frac{\vartheta_1'}{\vartheta_1}(u-\pi/3|\tau)=\frac{2}{\sqrt{3}}a+\frac{2}{\sqrt{3}}b\frac{R^2}{S},$$

which implies that

$$S'=\frac{2}{\sqrt{3}}\left(aS+bR^2\right). \tag{3.14}$$

Substituting (3.14) into (3.13), we deduce that

$$R'=-\frac{2}{\sqrt{3}}\left(aR+bS^2\right). \tag{3.15}$$

Next let

$$R = \sum_{j=0}^{\infty} r_j u^j$$

and

$$S = \sum_{j=0}^{\infty} s_j u^j.$$

Note that $r_0 = 0$ and $s_0 = 1$ and (3.15) and (3.14) imply that for $j \geq 1$,

$$r_j = -\frac{2}{\sqrt{3}j}\left(ar_{j-1} + b\sum_{\ell=0}^{j-1} s_\ell s_{j-1-\ell}\right)$$

and

$$s_j = \frac{2}{\sqrt{3}j}\left(as_{j-1} + b\sum_{\ell=0}^{j-1} r_\ell s_{r-1-\ell}\right).$$

These recurrences allow us to determine the series expansion of R and S. For example, the series expansion of R begins by

$$\begin{aligned} R(u|\tau) = &-\frac{2}{3}\sqrt{3}bu - \frac{2}{3}abu^2 + \left(-\frac{4}{9}\sqrt{3}a^2b\right)u^3 + \left(-\frac{10}{27}a^3b - \frac{8}{27}b^4\right)u^4 \\ &+ \left(-\frac{112}{405}\sqrt{3}ab^4 - \frac{44}{405}\sqrt{3}ba^4\right)u^5 + \left(-\frac{224}{405}a^2b^4 - \frac{28}{405}ba^5\right)u^6 \\ &+ \left(-\frac{256}{5103}\sqrt{3}b^7 - \frac{7232}{25515}\sqrt{3}a^3b^4 - \frac{344}{25515}\sqrt{3}a^6b\right)u^7 + \cdots, \end{aligned} \tag{3.16}$$

where $a = a(\tau)$ and $b = b(\tau)$. From (3.16), we derive the series expansion for $1/R^3$, which is

$$\begin{aligned} \frac{1}{R^3} = &-\frac{3\sqrt{3}}{8b^3}\frac{1}{u^3} + \frac{9}{8}\frac{a}{b^3}\frac{1}{u^2} + \left(\frac{1}{2} - \frac{9}{8}\frac{a^3}{b^3}\right) \\ &+ \left(\frac{9}{40}\frac{a^4\sqrt{3}}{b^3} - \frac{\sqrt{3}}{5}a\right)u + \left(\frac{27}{40}\frac{a^5}{b^3} - \frac{3}{5}a^2\right)u^2 \\ &+ \left(\frac{2\sqrt{3}}{7}a^3 - \frac{3\sqrt{3}a^6}{14b^3} - \frac{4\sqrt{3}}{63}b^3\right)u^3 + \cdots \end{aligned} \tag{3.17}$$

Now, $1/R^3$ is an elliptic function with pole of order 3 at $u = 0$. This implies that

$$\frac{1}{R^3} = -\frac{3\sqrt{3}}{16b^3}\left(\frac{\vartheta_1'}{\vartheta_1}(u|\tau)\right)'' + \frac{9}{8}\frac{a}{b^3}\left(\frac{\vartheta_1'}{\vartheta_1}(u|\tau)\right)' + \frac{9}{8}\left(\frac{a}{b^3}L_2 - \frac{a^3}{b^3}\right) + \frac{1}{2}, \tag{3.18}$$

where we have used (2.6).

Comparing the coefficients of u^{2k} in (3.18) with the use of (2.6) and (3.17), we arrive at the following identities:

Theorem 3.3. *Let L_{2j}, a and b be the functions defined in* (2.7), (3.1) *and* (3.2) *respectively. Then*

$$L_4(\tau) = a^4 \left(9 - 8\frac{b^3}{a^3}\right) \tag{3.19}$$

and

$$L_6(\tau) = a^6 \left(-27 + 36\frac{b^3}{a^3} - 8\frac{b^6}{a^6}\right). \tag{3.20}$$

4. Ramanujan's Differential Equations and Their Consequences

In [16], S. Ramanujan derived three differential equations involving $L_2(\tau), L_4(\tau)$ and $L_6(\tau)$. These are given by

$$x\frac{dL_2(\tau)}{dx} = \frac{L_2^2(\tau) - L_4(\tau)}{12} \tag{4.1}$$

$$x\frac{dL_4(\tau)}{dx} = \frac{L_2(\tau)L_4(\tau) - L_6(\tau)}{3} \tag{4.2}$$

$$x\frac{dL_6(\tau)}{dx} = \frac{L_2(\tau)L_6(\tau) - L_4^2(\tau)}{2}, \tag{4.3}$$

where $x = q^2 = e^{2\pi i\tau}$. There are two important identities that arise from (4.2) and (4.3). These are

$$3L_4^2(\tau)x\frac{dL_4(\tau)}{dx} - 2L_6(\tau)x\frac{dL_6(\tau)}{dx} = L_2(\tau)\left(L_4^3(\tau) - L_6^2(\tau)\right) \tag{4.4}$$

and

$$3L_6(\tau)x\frac{dL_4(\tau)}{dx} - 2L_4(\tau)\frac{dL_6(\tau)}{dx} = L_4^3(\tau) - L_6^2(\tau). \tag{4.5}$$

Ramanujan used (4.4) (see [16, (44)]) to deduce that

$$\Delta(\tau) = \frac{1}{1728}\left(L_4^3(\tau) - L_6^2(\tau)\right), \tag{4.6}$$

where

$$\Delta(\tau) = e^{2\pi i\tau}\prod_{k=1}^{\infty}(1 - e^{2k\pi i\tau})^{24}. \tag{4.7}$$

By (3.19) and (3.20), we find that the right hand side of (4.6) is

$$\frac{1}{1728}\left(L_4^3(\tau) - L_6^2(\tau)\right) = \frac{1}{3^3}b^9(a^3 - b^3). \tag{4.8}$$

Next, if

$$c = c(\tau) = 3q^{2/3}\prod_{k=1}^{\infty}\frac{(1 - q^{6k})^3}{1 - q^{2k}}, \tag{4.9}$$

then

$$c^3 b^9 = 3^3 \Delta(\tau). \tag{4.10}$$

Combining (4.6), (4.8) and (4.10), we deduce

Theorem 4.1. *Let a, b and c be functions defined in (3.1), (3.2) and (4.9). Then*

$$a^3 = b^3 + c^3, \tag{4.11}$$

or more explicitly,

$$\left(1 + 6\sum_{j=1}^{\infty}\left(\frac{j}{3}\right)\frac{q^{2j}}{1-q^{2j}}\right)^3 = \left(\prod_{k=1}^{\infty}\frac{(1-q^{2k})^3}{1-q^{6k}}\right)^3 + 27q^2\left(\prod_{k=1}^{\infty}\frac{(1-q^{6k})^3}{1-q^{2k}}\right)^3. \tag{4.12}$$

Identity (4.12) is equivalent to (1.3) using the identities established by the Borweins and Garvan [5].

Using (4.11), we now rewrite (3.19) and (3.20) as

Theorem 4.2. *Let L_{2j}, a, b and c be functions defined by (2.7), (3.1), (3.2) and (4.9) respectively. Then*

$$L_4(\tau) = a^4(1 + 8\alpha) \tag{4.13}$$

and

$$L_6(\tau) = a^6\left(1 - 20\alpha - 8\alpha^2\right), \tag{4.14}$$

where

$$\alpha = \alpha(\tau) = \frac{c^3(\tau)}{a^3(\tau)}. \tag{4.15}$$

Differentiating both sides of (4.13) and (4.14) with respect to $x = q^2$, we find that

$$x\frac{dL_4(\tau)}{dx} = 4a^3 x\frac{da}{dx}(1 + 8\alpha) + 8a^4 x\frac{d\alpha}{dx} \tag{4.16}$$

and

$$x\frac{dL_6(\tau)}{dx} = 6a^5 x\frac{da}{dx}\left(1 - 20\alpha - 8\alpha^2\right) + a^6(-20 - 16\alpha)\, x\frac{d\alpha}{dx}. \tag{4.17}$$

Using (4.13), (4.13), (4.16), (4.17) in (4.5), we deduce that

$$x\frac{d\alpha}{dx} = a^2\alpha(1 - \alpha). \tag{4.18}$$

Next, by using (4.13), (4.14), (4.16),(4.17) and (4.18), we deduce from (4.4) that

$$L_2(\tau) = 12a\alpha(1-\alpha)\frac{da}{d\alpha} + a^2(1 - 4\alpha). \tag{4.19}$$

Using (4.1), (4.19) and (4.18), we conclude that

$$\alpha(1-\alpha)\frac{d^2a}{d\alpha^2}+(1-2\alpha)\frac{da}{d\alpha}=\frac{2}{9}a.$$

This implies that

Theorem 4.3. *Let a be defined as in (3.1). Then*

$$a = {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha\right). \tag{4.20}$$

The proof of Theorem 4.3 sketched above can be found in [8].

5. Identities Associated with $L_4(3\tau)$ and $L_6(3\tau)$

In this section, we establish the parametrizations of $L_4(3\tau)$ and $L_6(3\tau)$ in terms of a and c. We will need the following analogue of (3.3):

Theorem 5.1. *Let $q=e^{\pi i\tau}$ with $Im\,\tau>0$ and a be defined as in (3.1). Then*

$$\begin{aligned}\frac{\vartheta_1'}{\vartheta_1}(u+\pi\tau|3\tau)-\frac{\vartheta_1'}{\vartheta_1}(u-\pi\tau|3\tau)+\frac{2i}{3}(2+a)\\ =2i\frac{\vartheta_1^2(u|3\tau)}{\vartheta_1(u+\pi\tau|3\tau)\vartheta_1(u-\pi\tau|3\tau)}\prod_{k=1}^{\infty}\frac{(1-q^{6k})^3}{1-q^{2k}}.\end{aligned} \tag{5.1}$$

Proof. The function

$$\frac{\vartheta_1'}{\vartheta_1}(u+\pi\tau|3\tau)-\frac{\vartheta_1'}{\vartheta_1}(u-\pi\tau|3\tau)$$

is elliptic with periods π and $3\pi\tau$ and has simple poles at $\pi\tau$ and $-\pi\tau$. The function

$$\frac{\vartheta_1^2(u|3\tau)}{\vartheta_1(u+\pi\tau|3\tau)\vartheta_1(u-\pi\tau|3\tau)}$$

is also elliptic with periods π and $3\pi\tau$ and simple poles at $\pi\tau$ and $-\pi\tau$. Therefore,

$$\frac{\vartheta_1'}{\vartheta_1}(u+\pi\tau|3\tau)-\frac{\vartheta_1'}{\vartheta_1}(u-\pi\tau|3\tau)+\kappa\frac{\vartheta_1^2(u|3\tau)}{\vartheta_1(u+\pi\tau|3\tau)\vartheta_1(u-\pi\tau|3\tau)}=\nu \tag{5.2}$$

for some κ and ν which are independent of u. Let $u=0$ in (5.2). We find, using (2.5), that

$$\nu=2\frac{\vartheta_1'}{\vartheta_1}(\pi\tau|3\tau)=-\frac{2i}{3}(a+2).$$

Therefore, we may rewrite (5.2) as

$$\begin{aligned}\vartheta_1'(u+\pi\tau|3\tau)\vartheta_1(u-\pi\tau|3\tau)-\vartheta_1'(u-\pi\tau|3\tau)\vartheta_1(u+\pi\tau|3\tau)+\kappa\vartheta_1^2(u|3\tau)\\ =-\frac{2i}{3}(a+2)\vartheta_1(u+\pi\tau|3\tau)\vartheta_1(u-\pi\tau|3\tau).\end{aligned} \tag{5.3}$$

Letting $u = \pi\tau$ in (5.3) and using (2.10) and (2.12), we deduce that

$$\kappa = \frac{\vartheta_1'(0|3\tau)\vartheta_1(2\pi\tau|3\tau)}{\vartheta_1^2(\pi\tau|3\tau)} = -2i\prod_{k=1}^{\infty}\frac{(1-q^{6k})^3}{1-q^{2k}}.$$

□

The following identity, which is similar to (3.6), follows from (5.1) using the argument as in the proof of (3.6).

Theorem 5.2. *Let $q = e^{\pi i\tau}$, with $Im\,\tau > 0$. Then*

$$\begin{aligned}&\vartheta_1^3(u|3\tau) - qe^{2iu}\vartheta_1^3(u+\pi\tau|3\tau) - qe^{-2iu}\vartheta_1^3(u-\pi\tau|3\tau)\\&\quad = 3\frac{a}{c}q^{2/3}\vartheta_1(u|3\tau)\vartheta_1(u+\pi\tau|3\tau)\vartheta_1(u-\pi\tau|3\tau),\end{aligned} \tag{5.4}$$

where c is defined in (4.9).

Identity (5.4) can be found in Liu's article [13, (1.13)], after applying [13, (5.1)]. The identity can also be found in Ramanujan's notebooks [1, p. 142, Entry 3].

Let

$$U = U(u|\tau) = q^{-1/3}e^{2iu/3}\frac{\vartheta_1(u|3\tau)}{\vartheta_1(u-\pi\tau|3\tau)} \tag{5.5}$$

and

$$V = V(u|\tau) = -e^{4iu/3}\frac{\vartheta_1(u+\pi\tau|3\tau)}{\vartheta_1(u-\pi\tau|3\tau)}. \tag{5.6}$$

By using U and V, we may rewrite (5.4) and (5.1) as

$$U^3 + V^3 + 3\frac{a}{c}UV = 1 \tag{5.7}$$

and

$$\frac{dV}{du} = -\frac{2i}{3}\left(aV + cU^2\right). \tag{5.8}$$

Using (5.7), we deduce that

$$U'\left(cU^2 + aV\right) = -V'(cV^2 + aU). \tag{5.9}$$

Using (5.8), we deduce from (5.9) that

$$\frac{dU}{du} = \frac{2i}{3}\left(aU + cV^2\right). \tag{5.10}$$

Using (5.10) and (5.8), we find that the first few terms of the power series expansion of U about 0 is given by

$$U = \frac{2ic}{3}u + \frac{2}{9}cau^2 - \frac{4}{27}ica^2u^3 + \left(-\frac{10}{243}ca^3 - \frac{8}{243}c^4\right)u^4 + \left(\frac{44}{3645}ica^4 + \frac{112}{3645}ic^4a\right)u^5 + \left(\frac{224}{10935}c^4a^2 + \frac{28}{10935}ca^5\right)u^6 + \left(-\frac{256}{137781}ic^7 - \frac{7232}{688905}ic^4a^3 - \frac{344}{688905}ica^6\right)u^7 + \cdots. \tag{5.11}$$

Using (5.11), we deduce that

$$\frac{1}{U^3} = \frac{27i}{8c^3}\frac{1}{u^3} - \frac{27a}{8c^3}\frac{1}{u^2} + \left(\frac{1}{2} - \frac{9}{8}\frac{a^3}{c^3}\right) + \left(-\frac{9i}{40}\frac{a^4}{c^3} + \frac{ai}{5}\right)u + \left(-\frac{9}{40}\frac{a^5}{c^3} + \frac{a^2}{5}\right)u^2 + \left(-\frac{ia^6}{14c^3} + \frac{2i}{21}ia^3 - \frac{4}{189}ic^3\right)u^3 + \cdots \tag{5.12}$$

Now, $1/U^3$ is an elliptic function with pole of order 3 at $u = 0$, with periods π and $3\pi\tau$. This implies that

$$\frac{1}{U^3} = -\frac{27i}{16c^3}\left(\frac{\vartheta_1'}{\vartheta}(u|3\tau)\right)'' + \frac{27}{8}\frac{a}{c^3}\left(\frac{\vartheta_1'}{\vartheta}(u|3\tau)\right)' + \frac{9}{8}\left(\frac{a}{c^3}L_2 - \frac{a^3}{c^3}\right) + \frac{1}{2}. \tag{5.13}$$

Comparing the coefficients of u^{2k} in (5.13) with the use of (2.6) and (5.12), we conclude that

Theorem 5.3. *Let L_{2j}, a and c be functions defined by* (2.7), (3.1) *and* (4.9). *Then*

$$L_4(3\tau) = a^4\left(1 - \frac{8}{9}\frac{c^3}{a^3}\right) \tag{5.14}$$

and

$$L_6(3\tau) = a^6\left(1 - \frac{4}{3}\frac{c^3}{a^3} + \frac{8}{27}\frac{c^6}{a^6}\right). \tag{5.15}$$

We end this section by observing that if we carry out the procedures illustrated in the proof of Theorem 4.3 using (3.19) and (3.20), we get

$$a = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \frac{b^3}{a^3}\right). \tag{5.16}$$

If we carry out the same procedures with (5.14) and (5.15), we get

$$a = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{c^3}{a^3}\right). \tag{5.17}$$

Identities (5.16) and (5.17) then imply that

$$\frac{c^3}{a^3} = 1 - \frac{b^3}{a^3},$$

giving us another proof of (4.11).

6. The Triplication Formula for $\alpha(\tau)$

From (4.7), (3.2), (4.9), we find, using (4.11) and (4.15), that

$$\Delta(\tau) = \frac{1}{3^3}c^3b^9 = \frac{a^{12}(\tau)}{3^3}\alpha(\tau)(1-\alpha(\tau))^3 \tag{6.1}$$

and

$$\Delta(3\tau) = \frac{1}{3^9}c^9b^3 = \frac{a^{12}(\tau)}{3^9}\alpha^3(\tau)(1-\alpha(\tau)). \tag{6.2}$$

Replacing τ by 3τ in (6.1), we find that

$$\Delta(3\tau) = \frac{1}{3^3} = \frac{a^{12}(3\tau)}{3^3}\beta(\tau)(1-\beta(\tau))^3, \tag{6.3}$$

where

$$\beta = \beta(\tau) = \alpha(3\tau). \tag{6.4}$$

Equating (6.2) and (6.3), we deduce that

$$a^{12}(\tau)\alpha^3(1-\alpha) = 3^6 a^{12}(3\tau)\beta(1-\beta)^3. \tag{6.5}$$

We also have two expressions for $L_4(3\tau)$, one from replacing τ by 3τ in (3.19) and the other from (5.14) and this implies that

$$a^{12}(\tau)\left(1-\frac{8}{9}\alpha\right)^3 = a^{12}(3\tau)\left(1+8\beta\right)^3. \tag{6.6}$$

Eliminating $a(\tau)$ and $a(3\tau)$ from (6.5) and (6.6), we conclude that

$$\alpha^3(1-\alpha)(1+8\beta)^3 = (9-8\alpha)^3\beta(1-\beta)^3. \tag{6.7}$$

Letting $s = (1-\alpha)^{1/3}$ and $t = \beta^{1/3}$ in (6.7), we obtain

$$(1-s^3)s(1+8t^3) = (1-t^3)t(1+8s^3),$$

which implies

$$(s-t)(s+2st-1+t)(s^2-2s^2t+4s^2t^2-2st^2+4st+s+t^2+t+1) = 0.$$

From the q-expansion of s and t, we obtain

$$t = \frac{1-s}{1+2s} \quad \text{and } s = \frac{1-t}{1+2t}.$$

In other words, we have

Theorem 6.1. *Let α and β be defined as in (4.15) and (6.4). Then*

$$\beta = \left(\frac{1-(1-\alpha)^{1/3}}{1+2(1-\alpha)^{1/3}}\right)^3 \tag{6.8}$$

and

$$\alpha = 1 - \left(\frac{1-\beta^{1/3}}{1+2\beta^{1/3}}\right)^3. \tag{6.9}$$

Substituting (6.8) and (6.9) into (6.6), we obtain

$$a(\tau) = \frac{3}{1+2(1-\alpha)^{1/3}} a(3\tau) \tag{6.10}$$

and

$$a(\tau) = (1+2\beta^{1/3})a(3\tau). \tag{6.11}$$

Next, note that

$$a(\tau) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \alpha\right) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-s^3\right)$$

and

$$a(3\tau) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \beta\right) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; t^3\right).$$

We can therefore translate (6.10) and (6.11) to the following transformation formulas for ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; z\right)$:

$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-s^3\right) = \frac{3}{1+2s}\, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \left(\frac{1-s}{1+2s}\right)^3\right) \tag{6.12}$$

and

$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-\left(\frac{1-t}{1+2t}\right)^3\right) = (1+2t)\, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; t^3\right). \tag{6.13}$$

By replacing s and t in (6.12) and (6.13) by a common variable r, we obtain

Theorem 6.2. *Let r be such that $0 < |r| < 1$. Then*

$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-r^3\right) = \frac{3}{1+2r}\, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \left(\frac{1-r}{1+2r}\right)^3\right) \tag{6.14}$$

and

$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; r^3\right) = \frac{1}{1+2r}\, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-\left(\frac{1-r}{1+2r}\right)^3\right). \tag{6.15}$$

We observe that (6.15) is (1.4) mentioned in Section 1.

From (6.10) and (6.11), we deduce the following identities:

Theorem 6.3. *Let a, b and c be functions defined by* (3.1), (3.2) *and* (4.9). *Then*

$$c(3\tau) = \frac{a(\tau) - a(3\tau)}{2}, \tag{6.16}$$

$$b(\tau) = \frac{3a(3\tau) - a(\tau)}{2}, \tag{6.17}$$

$$a(\tau) = 3c(3\tau) + b(\tau), \tag{6.18}$$

and

$$a(3\tau) = b(\tau) + c(3\tau). \tag{6.19}$$

Identities (6.16) and (6.17) allow us to express $c(\tau)$ and $b(\tau)$ in terms of Lambert series, for example,

$$b(\tau) = 1 + 9\sum_{j=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^{6k}}{1 - q^{6k}} - 3\sum_{j=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^{2k}}{1 - q^{2k}}.$$

Identities (6.18) and (6.19) yield two expressions of $a(\tau)$ in terms of infinite products, namely,

$$a(\tau) = \prod_{k=1}^{\infty} \frac{(1 - q^{2k})^3}{1 - q^{6k}} + 9q^2 \prod_{k=1}^{\infty} \frac{(1 - q^{18k})^3}{1 - q^{6k}} \tag{6.20}$$

$$= \prod_{k=1}^{\infty} \frac{(1 - q^{2k/3})^3}{1 - q^{2k}} + 3q^{2/3} \prod_{k=1}^{\infty} \frac{(1 - q^{6k})^3}{1 - q^{2k}}. \tag{6.21}$$

Using (4.11), (6.20), (6.21) and the definitions of b and c, we deduce that

$$\left(\prod_{k=1}^{\infty} \frac{(1 - q^{2k})^3}{1 - q^{6k}} + 9q^2 \prod_{k=1}^{\infty} \frac{(1 - q^{18k})^3}{1 - q^{6k}}\right)^3 = \left(\prod_{k=1}^{\infty} \frac{(1 - q^{2k})^3}{(1 - q^{6k})}\right)^3 + 27q^2 \left(\prod_{k=1}^{\infty} \frac{(1 - q^{6k})^3}{(1 - q^{2k})}\right)^3 \tag{6.22}$$

and

$$\left(\prod_{k=1}^{\infty} \frac{(1 - q^{2k/3})^3}{1 - q^{2k}} + 3q^{2/3} \prod_{k=1}^{\infty} \frac{(1 - q^{6k})^3}{1 - q^{2k}}\right)^3 = \left(\prod_{k=1}^{\infty} \frac{(1 - q^{2k})^3}{(1 - q^{6k})}\right)^3 + 27q^2 \left(\prod_{k=1}^{\infty} \frac{(1 - q^{6k})^3}{(1 - q^{2k})}\right)^3. \tag{6.23}$$

Identities (6.22) and (6.23), which are equivalent to (1.3) and (4.12), are analogues of (1.1).

Remark 6.1. For those who are familiar with the theory of modular forms, (6.23) can be written as

$$\left(1+\frac{\eta^3(9\tau)}{\eta^3(\tau)}\right)^3 = 1+27\frac{\eta^{12}(3\tau)}{\eta^{12}(\tau)},$$

which is a relation between the Hauptmodul for $\Gamma_0(9)$ and the Hauptmodul for $\Gamma_0(3)$.

7. Transformation for $a(\tau)$

Let the Dedekind η function be defined by

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{k=1}^{\infty}(1-e^{2\pi ik\tau}). \tag{7.1}$$

The function $\Delta(\tau)$ defined in (4.7) is

$$\Delta(\tau) = \eta^{24}(\tau). \tag{7.2}$$

Using (7.1), we rewrite (6.20) and (6.21) as

$$a(\tau) = \frac{\eta^3(\tau)}{\eta(3\tau)} + 9\frac{\eta^3(9\tau)}{\eta(3\tau)} \tag{7.3}$$

$$= \frac{\eta^3(\tau/3)}{\eta(\tau)} + 3\frac{\eta^3(3\tau)}{\eta(\tau)}. \tag{7.4}$$

It is known that

$$\Delta(-1/\tau) = \tau^{12}\Delta(\tau). \tag{7.5}$$

For a proof of (7.5), see [18] or [7, Theorem 6.10]. Using (7.5) and (7.2), we deduce that

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau). \tag{7.6}$$

By (7.3), (7.4) and (7.6), we find that

$$\begin{aligned} a(-1/(\sqrt{3}\tau)) &= \frac{\eta^3(-1/(3\sqrt{3}\tau))}{\eta(-1/(\sqrt{3}\tau))} + 3\frac{\eta^3(-3/(\sqrt{3}\tau))}{\eta(-1/(\sqrt{3}\tau))} \\ &= -i\tau\left(9\frac{\eta^3(9\tau/\sqrt{3})}{\eta(3\tau/\sqrt{3})} + \frac{\eta^3(\tau/\sqrt{3})}{\eta(3\tau/\sqrt{3})}\right) \\ &= -i\tau a(\tau/\sqrt{3}). \end{aligned} \tag{7.7}$$

By (5.16) and (7.7), we get

$$\tau = i\frac{{}_2F_1\left(1/3,2/3;1;1-\alpha(\tau/\sqrt{3})\right)}{{}_2F_1\left(1/3,2/3;1;\alpha(\tau/\sqrt{3})\right)}. \tag{7.8}$$

Observe that when $\alpha(\tau/\sqrt{3}) = 1/2$, then (7.8) implies that $\tau = i$. This yields the identity

$$\alpha(i/\sqrt{3}) = \frac{1}{2}.$$

Remark 7.1. 1. It is possible to derive (5.14) and (5.15) by using (7.6) and the transformation formulas for $L_4(\tau)$ and $L_6(\tau)$.

2. The transformation formula (7.7) for $a(\tau)$ can also be derived from (4.11) after applying (7.6) to $c(\tau)$ and $b(\tau)$. More precisely,

$$b\left(-\frac{1}{\sqrt{3}\tau}\right) = -i\tau c\left(\frac{\tau}{\sqrt{3}}\right)$$

implies that

$$a^3\left(-\frac{1}{\sqrt{3}\tau}\right) = (-i\tau)^3 a^3\left(\frac{\tau}{\sqrt{3}}\right),$$

which in turn implies (7.7).

8. Concluding Remarks

We have shown in this article that Ramanujan's theory of elliptic functions to the cubic base can be developed using identities associated with Jacobi's theta function $\vartheta_1(u|\tau)$. In our approach, $a(\tau)$ is defined in terms of Lambert series while $b(\tau)$ and $c(\tau)$ are infinite products and they arise naturally as coefficients of relations among certain elliptic functions.

Except for the paragraph before [4, Theorem 2.3] where $a(\tau)$, $b(\tau)$ and $c(\tau)$ are defined in terms of Lambert series, the functions $a(\tau)$, $b(\tau)$ and $c(\tau)$ are often defined in the literature by

$$a(\tau) = \sum_{m,n=-\infty}^{\infty} x^{m^2+mn+n^2},$$

$$b(\tau) = \sum_{m,n=-\infty}^{\infty} \omega^{m-n} x^{m^2+mn+n^2}$$

and

$$c(\tau) = \sum_{m,n=-\infty}^{\infty} x^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2},$$

where $x = q^2 = e^{2\pi i\tau}$. The derivation of $a(\tau), b(\tau)$ and $c(\tau)$ in terms of infinite products from its series version are then given by the Borweins and Garvan [5, Lemma 2.1, Proposition 2.2] (see [9] for the proof of the representation of $b(\tau)$ using the idea of [5]) and Z.G. Liu [13, Section 4].

The advantages of using Borweins' theta series as the starting point of Ramanujan's theory of elliptic functions to the cubic base is that the proofs of the identities (6.16)–(6.19) are simpler using the series representations of

$a(\tau), b(\tau)$ and $c(\tau)$. The identity (1.3) is then a consequence of (6.16)–(6.19) (see [5, Theorem 2.3]).

The disadvantages of using Borweins' theta series, on the other hand, is that it is harder to derive identities such as (4.13), (4.14), (6.8) and (5.16). For example, the proof of (5.16) by the Borweins [4] relied on a method similar to Gauss' work on the Arithmetic–Geometric Mean. A second proof of (5.16) was later given by B.C. Berndt, S.Bhargava and F.G. Garvan [3, Lemma 2.6] where the identity (6.13) (see [3, Corollary 2.4]),which is a special case of E. Goursat's transformation formula of the hypergeometric function (see [3, Theorem 2.3]), is used. In this article, we show that the transformation formula (6.13) is not required in the derivation of (5.16). Instead, (6.13) is a consequence of our work.

Finally, we say a few words about our functions R, S, U, V which are defined respectively in (3.10), (3.11), (5.5) and (5.6). There are two reasons why we believe that these functions are important. The first is that functions such as R and S, which can also be found in the master's thesis of S.T. Ng [14, Section 5.2], satisfy (3.12) and this relation is very similar to the relation [11, p. 171, (2)] satisfied by the Dixon cubic elliptic functions $\mathrm{sm}(u)$ and $\mathrm{cm}(u)$. We stress here that our functions R, S, U, V are not the Dixon functions. Another motivation which gives rise to these functions is Jacobi's elliptic function $\mathrm{sn}(u)$ (see [7, (7.2)]) which can be expressed in terms of

$$J(u) = -iq^{-1/2}e^{iu}\frac{\vartheta_1(u|2\tau)}{\vartheta_1(u-\pi\tau|2\tau)}.$$

Note the similarity between J and U defined in (5.5). We believe that with the discoveries of R, S, U, V, simpler proofs of some identities involving Jacobi's theta functions and Borweins' theta functions can probably be found.

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Conflict of interest The authors declare that there are no conflict of interests.

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